



Pergamon

Computers Math. Applic. Vol. 28, No. 1-3, pp. 281-289, 1994

Copyright©1994 Elsevier Science Ltd

Printed in Great Britain. All rights reserved

0898-1221/94 \$7.00 + 0.00

0898-1221(94)00116-2

Difference Equation Modelling of a Variable Structure System

R. B. POTTS

Applied Mathematics Department, The University of Adelaide
South Australia 5005

X.-H. YU

Department of Mathematics and Computing
University of Central Queensland
Rockhampton, QLD 4702, Australia

Abstract—Variable structure systems with sliding modes have been widely discussed and used in many different fields of applications. The precise behaviour at a switching surface is complicated because there the system is nonanalytic. A two-dimensional variable structure system is analysed in detail using a special discretisation with a difference equation model which is best in a particular sense. This reveals the occurrence and structure of discrete pseudo-sliding modes, which give insight to the corresponding sliding modes for the continuous system. Necessary and sufficient conditions are obtained for the occurrence of the pseudo-sliding modes and the analysis illustrated with graphs from numerical solutions.

Keywords—Difference equations, Variable structure systems, Discrete systems, Two-dimensional systems, Pseudo-sliding modes.

1. INTRODUCTION

Variable structure systems with sliding modes have been widely discussed [1] and used in many different fields of applications. Variable structure control is known to be robust to the variation of system parameters and external disturbances.

In essence, variable structure systems have been modelled mathematically by differential equations with discontinuous righthand sides [2]. The precise behaviour at a switching surface giving a sliding mode is complicated because there the system is nonanalytic.

The implementation of digital computer control requires the analysis of variable structure systems at discrete time intervals and this has generated new problems. For example, it has been shown [3] that the necessary and sufficient condition for the existence of a sliding mode for a continuous linear variable structure system is a necessary but not sufficient condition for the existence of a so-called pseudo-sliding mode for its sampled linear system. Some authors [4], in discussing the necessary and sufficient conditions for the existence of such pseudo-sliding modes, have introduced an approximation by freezing the inputs during a sampling period, but this masks some of the inherent properties.

In a sequence of recent papers [5–13], an analysis has been presented of a range of discrete variable structure systems exhibiting pseudo-sliding modes. The analysis has been modelled using difference equations which are ‘best’ approximations of the differential equations of the continuous systems in the special sense that for the linear part of the systems, the discrete solution points lie exactly on the continuous solution trajectories. The basic concept of such ‘best’ difference equations is described in references [14,15].

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

It is the purpose of the present paper to analyse a two-dimensional discrete variable structure system using a complete discretisation using 'best' difference equation approximations to the corresponding differential equations of the continuous system. The discretisation with these special difference equations enables a complete analysis using only sequences of position signals. Necessary and sufficient conditions for the occurrence of discrete pseudo-sliding modes are derived.

2. CONTINUOUS TWO-DIMENSIONAL SYSTEM

Consider the two-dimensional variable structure system [16] described by

$$\dot{x}_1(t) = x_2(t), \quad (1)$$

$$\dot{x}_2(t) = -fx_2 - u, \quad (2)$$

with control law given by

$$u = \psi x_1, \quad (3)$$

with ψ defined by

$$\psi = \begin{cases} a, & \text{if } x_1 s(x) > 0, \\ -a, & \text{if } x_1 s(x) < 0, \end{cases} \quad (4)$$

where

$$s = s(\mathbf{x}) = cx_1 + x_2. \quad (5)$$

Without loss of generality, the constants a, c are taken to be positive. The constant f may be positive or negative. The switching line $s(\mathbf{x}) = 0$ is the possible sliding line.

The characteristic equation corresponding to the second order system is

$$m^2 + fm \pm a = 0, \quad (6)$$

with roots

$$m_1 = -\alpha + i\beta, \quad m_2 = -\alpha - i\beta, \quad (7)$$

where

$$\alpha = \frac{f}{2}, \quad (8)$$

$$\beta = \frac{1}{2} \sqrt{\pm 4a - f^2}. \quad (9)$$

The constant β is real or pure imaginary.

The equation of the orbits in the x_1, \dot{x}_1 phase plane is

$$V(t) = \dot{x}_1^2 + f\dot{x}_1 x_1 \pm ax_1^2, \quad (10)$$

$$= e^{-ft} V(0). \quad (11)$$

It is well known [17] that the necessary and sufficient conditions for

$$s = cx_1 + \dot{x}_1 = 0, \quad (12)$$

to be a sliding line are

$$\lim_{s \rightarrow +0} \dot{s} \leq 0, \quad \lim_{s \rightarrow -0} \dot{s} \geq 0. \quad (13)$$

Since

$$\lim_{s \rightarrow +0} \dot{s} = \lim_{s \rightarrow -0} \dot{s} = x_1(cf - c^2 \mp a), \quad (14)$$

there exists a stable sliding mode provided that

$$c^2 - a \leq cf \leq c^2 + a. \quad (15)$$

3. DISCRETE TWO-DIMENSIONAL SYSTEM

We now consider a discretisation of the system by letting $x(k)$ denote x_1 evaluated at time kh , where h is a discrete constant time interval, not necessarily 'small.' The 'best' discretisation of the system is given by the difference equation

$$x(k+2) - [e^{m_1 h} + e^{m_2 h}]x(k+1) + e^{(m_1+m_2)h}x(k) = 0, \quad (16)$$

with m_1 and m_2 defined by (7) and in which the following choice in determining β from (9) is made:

$$\text{use top sign if } x(k)[x(k+1) - e^{-ch}x(k)] \geq 0 \quad (17)$$

$$\text{use bottom sign if } x(k)[x(k+1) - e^{-ch}x(k)] < 0. \quad (18)$$

This discretisation is the 'best' in the sense that the solution of the difference equation (16) gives discrete points exactly on the solution curve of the differential equation (1,2), regardless of the magnitude of h [14].

In the $x(k), x(k+1)$ discrete phase plane, the top sign is used for regions I and III in Figure 1 and the bottom sign for regions II and IV. The variable structure corresponds to the switching line $s(k) = 0$ where

$$s(k) = x(k+1) - e^{-ch}x(k). \quad (19)$$

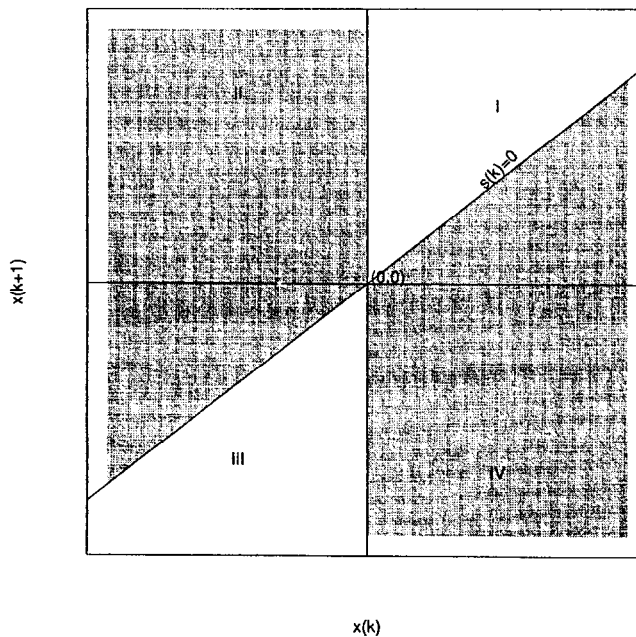


Figure 1. The $x(k), x(k+1)$ phase plane is divided into regions corresponding to the sign of $x(k)s(k)$, where $s(k) = x(k+1) - e^{-ch}x(k) = 0$ is the switching line. For the regions I and III, the top sign in (9) has to be taken; for the shaded regions II and IV, the bottom sign.

The second order difference equation (16) can be written in canonical matrix form as

$$\begin{bmatrix} x(k+1) \\ x(k+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -e^{-2\alpha h} & 2e^{-\alpha h} \cos(\beta h) \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix}. \quad (20)$$

The orbits in the discrete phase plane can be constructed from curves with equations

$$W(k) = [x(k+1) - e^{m_1 h}x(k)][x(k+1) - e^{m_2 h}x(k)], \quad (21)$$

$$= x(k+1)^2 - 2e^{-\alpha h} \cos(\beta h)x(k+1)x(k) + e^{-2\alpha h}x(k)^2, \quad (22)$$

$$= e^{-fkh}W(0). \quad (23)$$

The nature of these curves, and hence of the orbits, depends on the values of the parameters, and the following cases arise:

1. for the top sign
 - (a) $0 < f < 2\sqrt{a}$ contracting ellipses,
 - (b) $0 > f > -2\sqrt{a}$ expanding ellipses,
 - (c) $f > 2\sqrt{a}$, $m_1 < m_2 < 0$ contracting hyperbolas,
 - (d) $f < -2\sqrt{a}$, $0 < m_1 < m_2$ expanding hyperbolas,
2. for the bottom sign
 - (a) $f > 0$, $m_1 < 0 < m_2$ contracting hyperbolas,
 - (b) $f < 0$, $m_1 < 0 < m_2$ expanding hyperbolas.

The descriptions contracting and expanding refer to the effect of the factor $\exp(-fkh)$ in (23) which forces $W(k) \rightarrow 0$ or $W(k) \rightarrow \infty$ as $k \rightarrow \infty$, according as the friction f is positive or negative.

The construction of the orbits in the $x(k), x(k+1)$ phase plane proceeds in the usual way from an initial point by first projecting horizontally to the line $x(k+1) = x(k)$, then vertically to the next appropriate $W(k)$ curve, then horizontally and vertically in staircase fashion. Figure 2 represents the construction of the discrete points on an orbit for Case 1.(a) when the $W(k)$ curves are contracting ellipses and Figure 3 to Case 2.(a) when the curves are contracting hyperbolas.

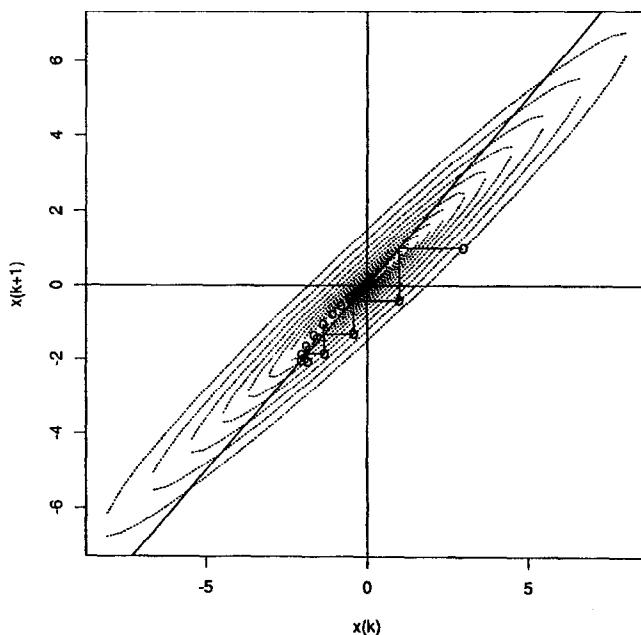


Figure 2. Construction of an orbit using the values $c = 1$, $a = 9$, $f = 4$, $h = 0.1$, $x(0) = 3$, $x(1) = 1$, and the top sign in (9). The contracting ellipse and the line $x(k+1) = x(k)$ are used to determine points on the orbit. The discretisation is exact in the sense that the points marked with circle are successive points on an exact orbit of the differential equation.

When the orbits are obtained from hyperbola-like curves, corresponding to β being pure imaginary, so that m_1 and m_2 are real, the equations of the asymptotes are, from (21),

$$x(k+1) = e^{m_1 h} x(k), \quad (24)$$

$$x(k+1) = e^{m_2 h} x(k). \quad (25)$$

These asymptotes play a role in determining the occurrence of a pseudo-sliding mode. In Figure 3, it is clear that the two asymptotes lie on either side of the line $x(k+1) = x(k)$, corresponding to $m_1 < 0 < m_2$.

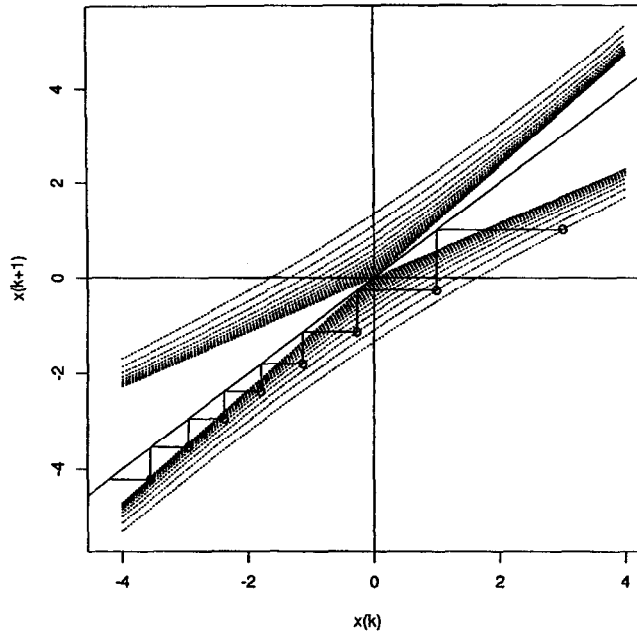


Figure 3. Construction of an orbit using the same parameter values as in Figure 2 but now with the bottom sign in (9). The contracting curves are now hyperbolas.

4. CONDITIONS FOR PSEUDO-SLIDING MODE

A necessary condition for the occurrence of a stable pseudo-sliding mode is that the trajectories in the $x(k), x(k+1)$ phase plane approach the switching line $s = 0$ from both sides. There are two possibilities. For a trajectory from region I (see Figure 1) for which $x(k)$ and $s(k)$ are both positive, the requirement that the trajectory crosses the switching line to region IV implies that in the limiting case, if $s(k) = 0$, then the next value evaluated, $s(k+1)$, must be negative. In the same way, from region III for which $x(k)$ and $s(k)$ are both negative, the next value $s(k+1)$, evaluated after an $s(k) = 0$, must be positive. In both evaluations, the top sign in (9) must be used. In summary, if $s(k) = 0$, and $x(k)$ and $s(k+1)$ are of opposite sign, then the top sign must be used. In the second case, the requirement that the trajectory crosses the switching line from region IV to I or II to III implies that $s(k+1)$ and $x(k)$ must be of the same sign when the bottom sign of (9) is used.

The consequence of this argument leads to the formulation of the necessary condition. From (19), it follows that, if $s(k) = 0$, then

$$\begin{bmatrix} x(k) & x(k+1) \end{bmatrix}^T = \begin{bmatrix} 1 & e^{-ch} \end{bmatrix}^T x(k), \quad (26)$$

and that

$$s(k+1) = \begin{bmatrix} -e^{-ch} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -e^{-2\alpha h} & 2e^{-\alpha h} \cos(\beta h) \end{bmatrix} \begin{bmatrix} 1 \\ e^{-ch} \end{bmatrix} x(k). \quad (27)$$

This simplifies to

$$s(k+1) = \left[2e^{-(\alpha+c)h} \cos(\beta h) - e^{-2\alpha h} - e^{-2ch} \right] x(k). \quad (28)$$

The necessary condition for the existence of a stable pseudo-sliding mode, namely that $x(k)$ and $s(k+1)$ are of opposite signs for the top sign in (9) and of the same sign for the bottom sign, is that

$$E = 2e^{-(\alpha+c)h} \cos(\beta h) - e^{-2\alpha h} - e^{-2ch}, \quad (29)$$

should be negative for the top and positive for the bottom sign. Since

$$E = -\left[e^{-\alpha h} \cos(\beta h) - e^{-ch} \right]^2 - e^{-2\alpha h} \sin^2(\beta h), \quad (30)$$

it is always negative if β is real, and if β is pure imaginary

$$E = -e^{-2ch} \left[e^{(c-\alpha-i\beta)h} - 1 \right] \left[e^{(c-\alpha+i\beta)h} - 1 \right], \quad (31)$$

$$= -e^{-2ch} \left[e^{(2c-f-\sqrt{f^2+4a})/2} - 1 \right] \left[e^{(2c-f+\sqrt{f^2+4a})/2} - 1 \right]. \quad (32)$$

It follows after some algebra that necessary conditions for a sliding mode are

$$c^2 - a < fc < c^2 + a, \quad (33)$$

in agreement with (15) for the continuous case.

A sufficient condition can be determined from the following argument. It is clear that the condition must involve a restriction on the magnitude of h , for this controls the sizes of the steps along a trajectory. What is required is that when a trajectory is being stepped out, a step across the switching line must be sufficiently small so that the region which forces the return in the direction to the switching line is not over-stepped. This region is bounded by the switching line $s(k) = 0$ and the asymptote $r(k) = 0$, where

$$r(k) = x(k+1) - \exp(m_1 h) x(k). \quad (34)$$

From (7) and (9),

$$m_1 = -\alpha - \gamma, \quad (35)$$

where

$$\gamma = (1/2)\sqrt{4a + f^2}. \quad (36)$$

The asymptote further divides regions II and IV into subregions as illustrated in Figure 4.

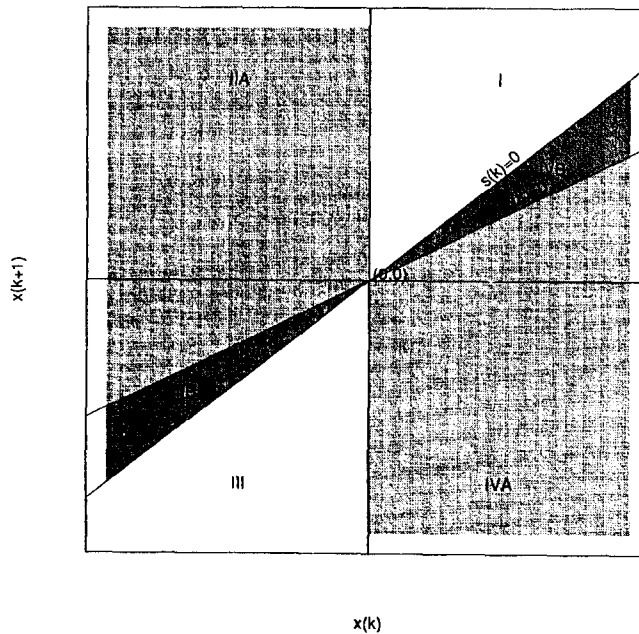


Figure 4. The phase plane in Figure 1 is further subdivided by the asymptote $r(k) = 0$. The heavily shaded regions IIB and IVB correspond to regions where the trajectories move towards the switching line $s(k) = 0$.

The limiting case occurs when a step is from the switching line to the asymptote, that is, from $s(k) = 0$ to $r(k+1) = 0$. As in the derivation of (27),

$$r(k+1) = \begin{bmatrix} -e^{-(\alpha+\gamma)h} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -e^{-2\alpha h} & 2e^{-\alpha h} \cos(\beta h) \end{bmatrix} \begin{bmatrix} 1 \\ e^{-ch} \end{bmatrix} x(k), \quad (37)$$

which leads to

$$r(k+1) = e^{-(c+\alpha)h} \left[2 \cos(\beta h) - e^{-\gamma h} - e^{(c-\alpha)h} \right] x(k). \quad (38)$$

The condition $r(k+1) = 0$ places an upper bound H on the value of h given by

$$2 \cos(\beta H) - e^{-\gamma H} - e^{(c-\alpha)H} = 0. \quad (39)$$

The top sign is used in calculating β , because $r(k+1)$ is calculated from a previous point in region I or III for a trajectory just straddling region IVB or IIB.

5. NUMERICAL RESULTS

The constant c in (17) and (18) is positive and rescaling the time interval h enables c to be taken equal to unity for numerical calculations without loss of generality. To explain the scaling procedure, suppose that in the given system a sliding mode with a time constant of 0.5 seconds is required; then this is simply the unit in which h is measured.

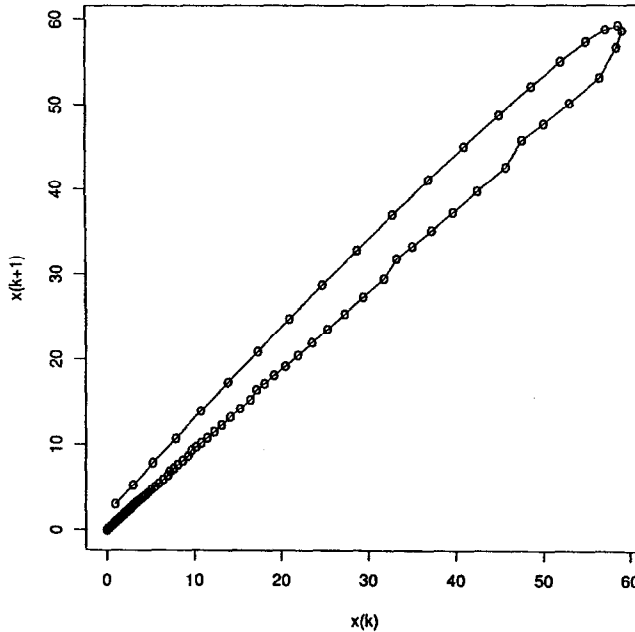


Figure 5. A stable pseudo-sliding mode with the following values: $c = 1$, $a = 9$, $f = -3.49$, $h = 0.046$, $x(0) = 1$, $x(1) = 3$. The value of h is just less than the upper bound $H \approx 0.046536$. Under the effect of the negative friction, the orbit first moves away from the origin but then returns and chatters down the switching line (not drawn, but with equation $x(k+1) = \exp(-h)x(k)$).

Figure 5 illustrates a stable pseudo-sliding mode with $c = 1$, $a = 9$, $f = -3.49$, $h = 0.046$. The value of f corresponds to 'moderate' negative friction but is still in the range $-8 < f < 10$ for a stable sliding-mode for the continuous system (see (33)). The value of the step-size h is less than the upper bound $H \approx 0.046536$ calculated from (39). The chattering down the switching line is clearly evident.

Figure 6 illustrates a stable system without a pseudo-sliding mode. The data is as for Figure 5 except that $h = 0.05$, which just exceeds the upper bound. The orbit veers away then returns to circuit the origin. This behaviour is repeated indefinitely, as the orbit converges to the origin.

Finally, Figure 7 illustrates a stable pseudo-sliding mode with positive friction. To take an exaggerated example, a large value $h = 0.5$ is assumed; the chattering down the switching line is very marked.

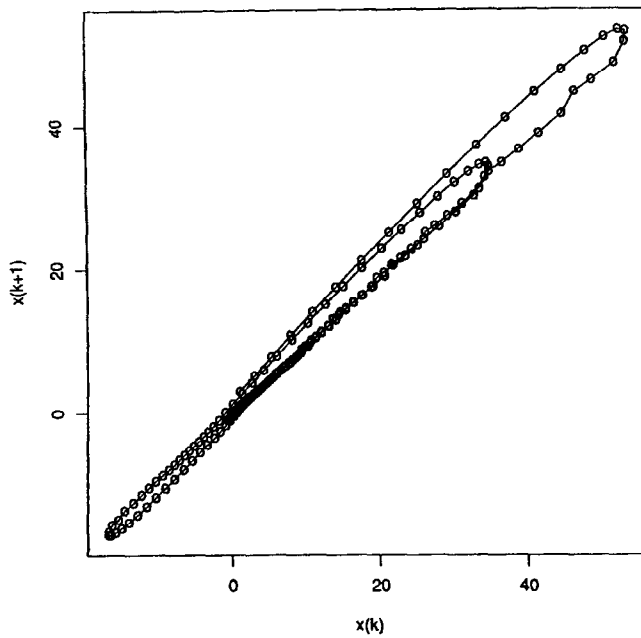


Figure 6. A stable system without a pseudo-sliding mode. The data is the same as for Figure 5 except that $h = 0.05$, exceeding the upper bound. The orbit first goes away to a point near $(50, 50)$, then returns towards the origin along the switching line but without proper sliding as it overshoots to near the point $(-15, -15)$. It then goes to near $(40, 40)$, and loops around the origin indefinitely, but converging to it.

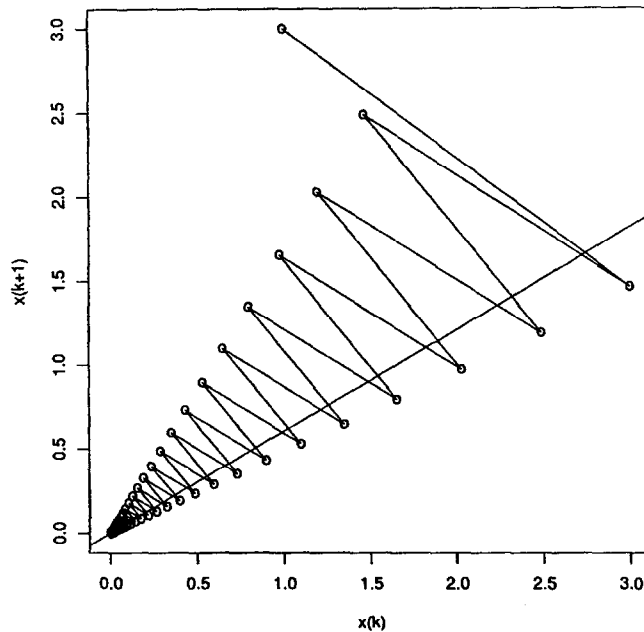


Figure 7. A stable pseudo-sliding mode with the following values: $c = 1$, $a = 9$, $f = 7$, $h = 0.5$, $x(0) = 1$, $x(1) = 3$. The friction is now positive. Even with a large stepsize h , the orbit chatters down the switching line to the origin.

6. DISCUSSION

The fact that the variable structure system considered in this paper is, apart from the switching logic, a linear system with constant coefficients, has enabled a special discretisation using 'best' difference equations such that the behaviour of the discrete system is not compounded by any approximate integration of the differential equation. This difference equation modelling has

enabled necessary and sufficient conditions for the occurrence of a pseudo-sliding mode to be determined, and these have been substantiated by numerical results. Not only has the well-known theory of continuous variable structure systems been confirmed in the limit as the discrete stepsize approaches zero, but also considerable insight has been given to the occurrence and structure of sliding modes.

REFERENCES

1. V.I. Utkin, Discontinuous control systems: State of art in the theory and applications, *10th IFAC World Congress on Automatic Control, Munich* **1**, 75–94 (1987).
2. A.F. Filippov, *Differential Equations with Discontinuous Righthand*, Kluwer, Dordrecht, (1988).
3. C. Milosavljevic, General conditions for the existence of a quasi-sliding mode on the switching hyperplane in discrete variable structure systems, *Autom. Remote Control* **46**, 307–325 (1985).
4. S.Z. Sarpturk, Y. Istefanopulos and O. Kaynak, On the stability of discrete-time sliding mode control systems, *IEEE Trans. Autom. Control* **AC-32**, 930–932 (1989).
5. R.B. Potts and X. Yu, Three-dimensional discrete variable structure system with pseudo-sliding mode, *Proceedings of the University of Adelaide Special Symposium*, pp. 329–342, (1990).
6. X. Yu and R.B. Potts, Discrete VSS control for a cylindrical robot, *Proceedings of 1st IEEE International Workshop on Variable Structure Systems and their Applications*, Sarajevo, Yugoslavia, pp. 98–109, (1990).
7. R.B. Potts and X. Yu, Discrete variable structure system with pseudo-sliding mode, *J. Austral. Math. Soc. Ser. B* **32**, 365–376 (1991).
8. X. Yu and R.B. Potts, A class of discrete variable structure systems, *Proceedings of 30th IEEE Conference on Decision and Control*, Brighton, UK, pp. 1367–1372, (1991).
9. X. Yu and R.B. Potts, Computer-controlled variable structure systems, *J. Austral. Math. Soc. Ser. B* **34**, 1–17 (1992).
10. X. Yu and R.B. Potts, Analysis of discrete variable structure systems with pseudo-sliding modes, *Int. J. Syst. Sci.* **23**, 503–516 (1992).
11. X. Yu, Chaos in discrete variable structure systems, *Proceedings of 31st IEEE Conference on Decision and Control, Arizona, USA* **2**, 1862–1863 (1992).
12. X. Yu, Digital sliding mode control of linear systems, *Proceedings of 2nd IEEE International Workshop on Variable Structure and Lyapunov Control of Uncertain Dynamical Systems*, Sheffield, UK, pp. 191–196, (1992).
13. X. Yu, Discrete variable structure control systems, *Int. J. Syst. Sci.* (in press).
14. R.B. Potts, Differential and difference equations, *Amer. Math. Monthly* **89**, 402–407 (1982).
15. R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, M. Decker, New York, (1992).
16. V.I. Utkin, Variable structure systems with sliding modes, *IEEE Trans. Autom. Control* **AC-22**, 212–222 (1977).
17. E.A. Barbashin, *Introduction to the Theory of Stability*, Wolters Noord-hoff, Groningen, (1970).